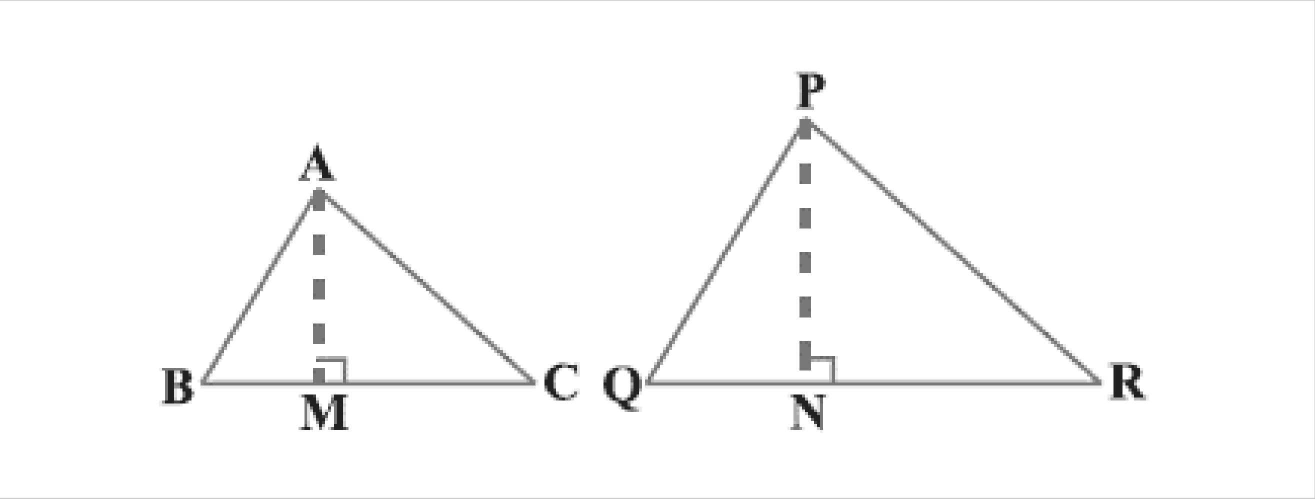
TRIANGLES

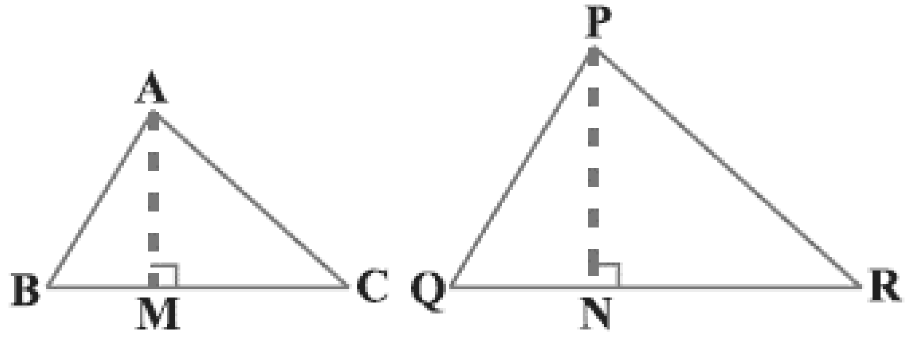
MODULE 4

AREAS OF SIMILAR TRIANGLES

The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.



PROOF OF THEOREM FOR AREAS OF SIMILAR TRIANGLES



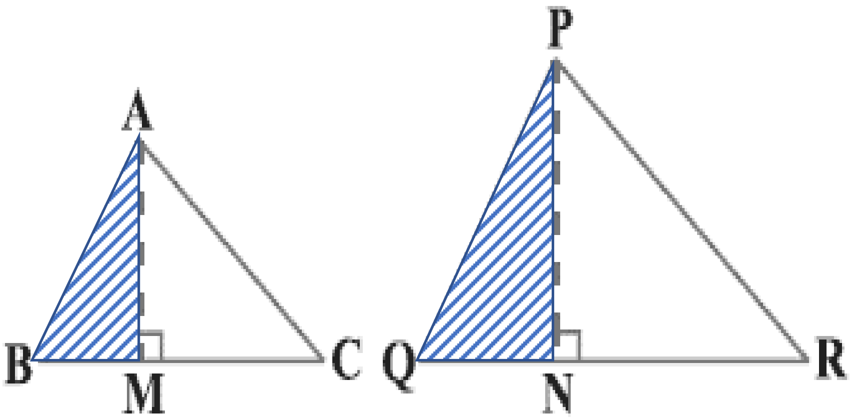
Consider two triangles, ΔABC and ΔPQR

- **Given:** $\Delta ABC \sim \Delta \mathrm{PQR}$
- **To prove:**
$$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{AC}{PR}\right)^2$$
- **Construction**: Draw altitudes AM and PN of the triangles ABC and PQR from vertices A and P respectively

$$ar(\Delta ABC) = \frac{1}{2} \times BC \times AM$$

$$ar(\Delta PQR) = \frac{1}{2} \times QR \times PN$$

$$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \frac{\frac{1}{2} \times BC \times AM}{\frac{1}{2} \times QR \times PN} = \frac{BC \times AM}{QR \times PN} \quad \text{----------- (1)}$$



Since $\Delta ABC \sim \Delta PQR$, their corresponding angles are equal. $\angle B = \angle Q$

Also, $\angle M = \angle N = 90°$

Therefore $\Delta ABM \sim \Delta PQN$ by AA similarity criterion

$$\frac{AM}{PN} = \frac{AB}{PQ} \quad \text{----------- (2)}$$

Since $\Delta ABC \sim \Delta PQR$, their corresponding sides are proportional

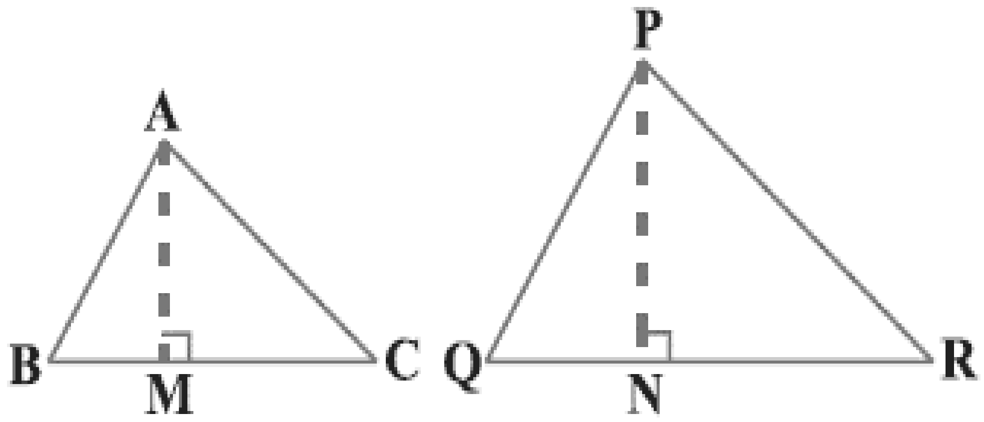
$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} \quad \text{----------- (3)}$$

From (1) and (3)

$$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \frac{AB}{PQ} \times \frac{AM}{PN}$$

From (2)

$$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \frac{AB}{PQ} \text{ x } \frac{AB}{PQ} = \left(\frac{AB}{PQ}\right)^2$$



From (3), we get

$$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{AC}{PR}\right)^2$$

Hence we can conclude that the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

SOLVED EXAMPLE

ΔABC is right angled at C and CD is perpendicular to AB, prove that BC^2 x AD= AC^2 x BD

Given, ΔABC is right angled at C and CD is perpendicular to AB

To prove, BC^2 x AD= AC^2 x BD

Proof: Consider ΔACD and ΔDCB

Let $\angle A = x \Longrightarrow \angle B$ = 90° - x , as ΔACB is right angled

In ΔADC and ΔCDB,

$\angle ADC$ = $\angle CDB = 90°$

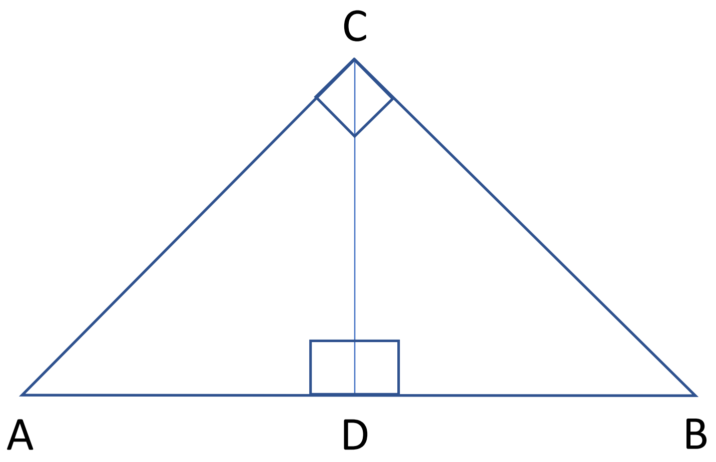
$\angle A$ = $\angle DCB = x$

By AA Similarity, ΔACD ~ ΔDCB

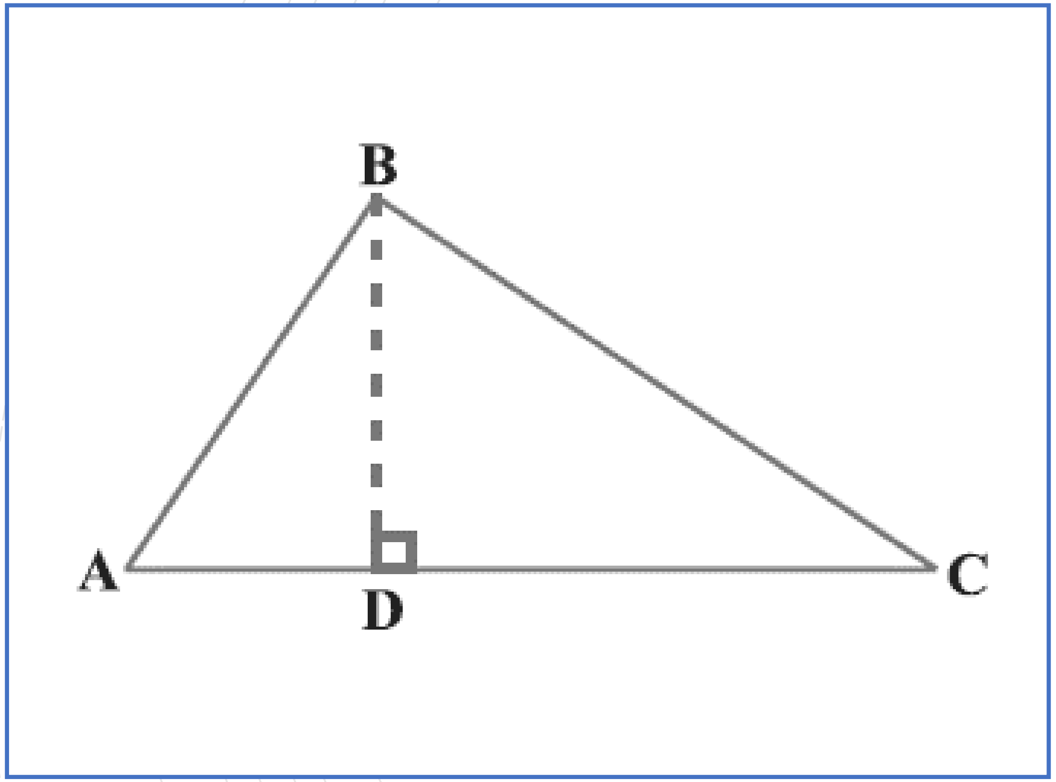
$\Longrightarrow \frac{ar(\Delta ACD)}{ar(\Delta CBD)} = \frac{AC^2}{BC^2} \Longrightarrow \frac{\frac{1}{2}\times AD\times CD}{\frac{1}{2}\times BD\times CD} = \frac{AC^2}{BC^2}$

$\Longrightarrow \frac{AD}{DB} = \frac{AC^2}{BC^2}$

$\Longrightarrow BC^2$ x AD= AC^2 x BD



SIMILARITY OF TRIANGLES IN A RIGHT TRIANGLE



- Given: ΔABC is a right triangle, $\angle ABC = 90°$
- To prove:

  $\Delta ADB \sim \Delta ABC,$

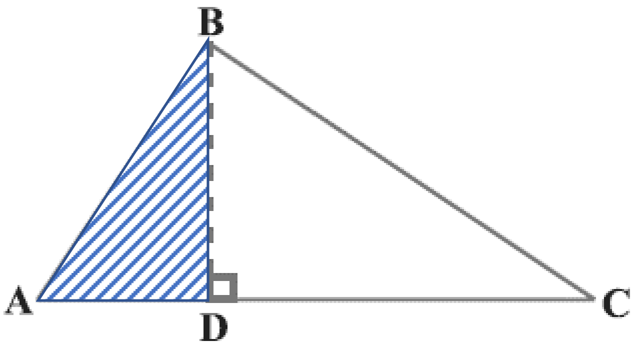
  $\Delta ABC \sim \Delta BDC$ and

  $\Delta ADB \sim \Delta BDC$
- Construction: Draw BD perpendicular to hypotenuse AC

Consider ΔABC and ΔADB

$\angle A$ is common and $\angle ADB = \angle ABC = 90°$

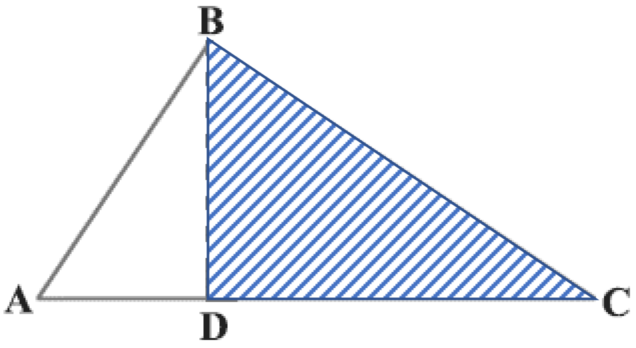
Therefore $\Delta ABC \sim \Delta ADB$ by AA similarity criterion



Consider ΔABC and ΔBDC

$\angle C$ is common and $\angle BDC = \angle ABC = 90°$

Therefore $\Delta ABC \sim \Delta BDC$ by AA similarity criterion

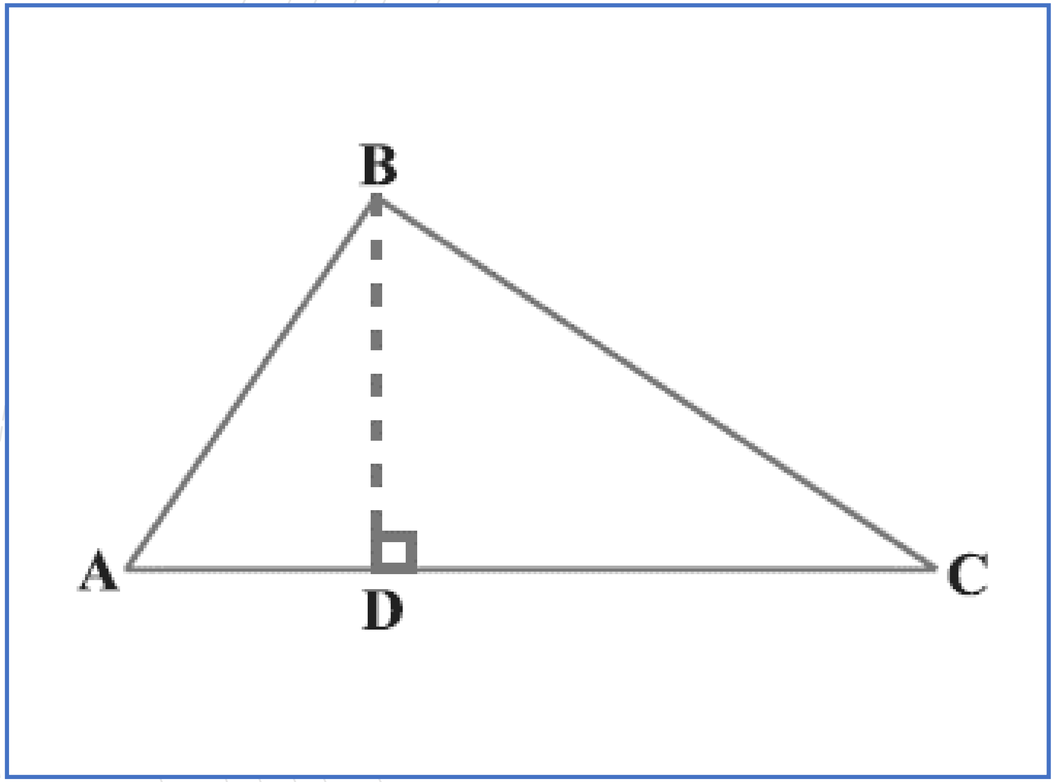


If one triangle is similar to another triangle and this second triangle is similar to a third triangle, then the first triangle is similar to the third triangle

$\Longrightarrow \Delta ADB \sim \Delta BDC$

If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse then triangles on both sides of the perpendicular are similar to the whole triangle and to each other.

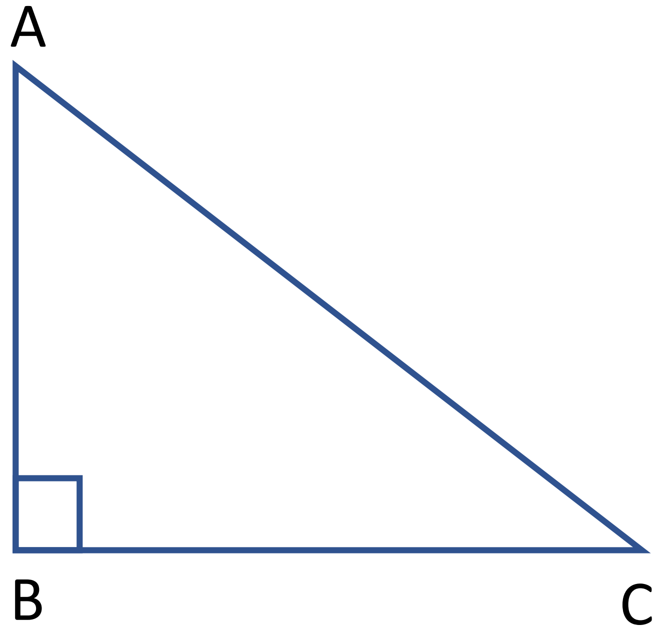
PROOF OF PYTHAGORAS THEOREM USING SIMILARITY



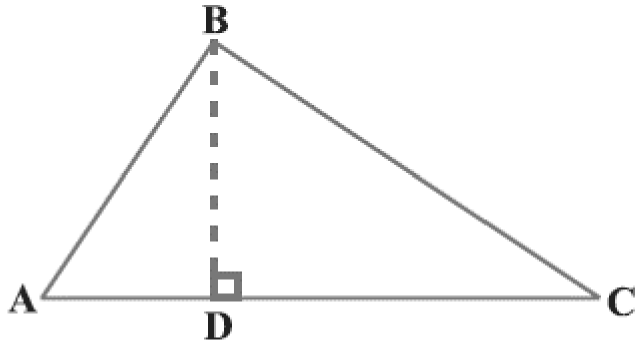
- Given: ΔABC is a right triangle, $\angle ABC = 90°$
- To prove: $AC^2 = AB^2 + BC^2$
- Construction: Draw BD perpendicular to hypotenuse AC

PYTHAGORAS THEOREM

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.



$$AC^2 = AB^2 + BC^2$$



Given: ΔABC is a right triangle, right-angled at B

To Prove: $AC^2 = AB^2 + BC^2$

Construction: BD ⊥ AC

Proof:

Since $\Delta ADB \sim \Delta ABC$, their sides are proportional

$$\frac{AD}{AB} = \frac{AB}{AC}$$

By cross multiplication, $AD \times AC = AB^2$ ----------- (1)

Since $\Delta BDC \sim \Delta ABC$, their sides are proportional

$$\frac{CD}{BC} = \frac{BC}{AC}$$

By cross multiplication, $CD \times AC = BC^2$ ----------- (2)

Adding (1) and (2)

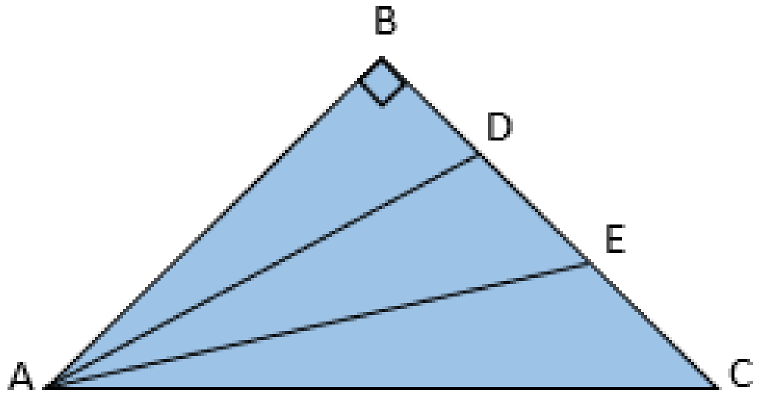
$$AD \times AC + CD \times AC = AB^2 + BC^2$$
$$AC(AD + CD) = AB^2 + BC^2$$
$$AC \times AC = AB^2 + BC^2$$
$$AC^2 = AB^2 + BC^2$$

SOLVED EXAMPLE

In figure, ΔABC is right angled at B. Side BC is trisected at points D and E, Prove that $8AE^2$ = $3AC^2$ + $5AD^2$

Given, ΔABC is right angled at B. Side BC is trisected at points D and E

To Prove: $8AE^2$ = $3AC^2$ + $5AD^2$



Proof:

D and E are the points of trisection of BC.

BD = $\frac{1}{3}$BC and BE= $\frac{2}{3}$BC ------------ (i)

In right-angled ΔABD, Using Pythagoras theorem,

$AD^2 = AB^2 + BD^2$ ------------ (ii)

In ΔABE,

$AE^2 = AB^2 + BE^2$ ------------- (iii)

In ΔABC,

$AC^2 = AB^2 + BC^2$ -------------- (iv)

From (ii) and (iii), we have

AD^2 - AE^2= BD^2 - BE^2

$\Longrightarrow AD^2$ - AE^2= $(\frac{1}{3}\text{BC})^2$- $(\frac{2}{3}\text{BC})^2$

$\Longrightarrow AD^2$ - AE^2= $\frac{1}{9}\text{BC}^2$ - $\frac{4}{9}\text{BC}^2$ = - $\frac{3}{9}\text{BC}^2$

$\Longrightarrow AE^2$ - AD^2= $\frac{1}{3}\text{BC}^2$ ---------- (v)

From (iii) and (iv), we have

AC^2 - AE^2 = BC^2 - BE^2

= BC^2 - $\frac{4}{9}\text{BC}^2$

$\Longrightarrow AC^2$ - AE^2= $\frac{5}{9}\text{BC}^2$ ----------- (vi)

From (v) and (vi), we get

AC^2 - AE^2 = $\frac{5}{3}(AE^2$ - $AD^2)$

$\Longrightarrow$ 3AC^2 - 3AE^2 = 5AE^2 - 5AD^2

$\Longrightarrow$ 8AE^2 = 5AD^2+ 3AC^2

THANK YOU