STRAIGHT LINES

MODULE 1/3

In this module we will study about

- Some Important formula from previous classes
- Slope of a straight line
  - If the angle made by the line with x-axis is known.
  - When the coordinates of any two points on the line is given.
- Angle between two lines
- Conditions for parallel and perpendicular lines
- Collinearity of three points
- Some example problems
- Problems for practice

Recall of important formula

1. Distance between two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ is
   $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

2. Section formula

   The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio m: n are $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

3. In particular when m = n, the coordinate of the midpoint of the line segment joining the points (x_1,y_1) and (x_2,y_2) are $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$

4. Area of the triangle whose vertices are (x_1,y_1), (x_2,y_2) and (x_3,y_3) is
   $\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

Remark: If the area of the triangle ABC is zero, then the three points A,B and C lie on a line, i.e, they are collinear

Slope of a straight line

Any line (l) in the coordinate plane forms two angles with the x-axis which are supplementary. If **θ** be the angle made by the line 'l' with the positive direction of x- axis and measured anti-clockwise is called the inclination of the line and obviously $0 \leq \theta \leq 180$.

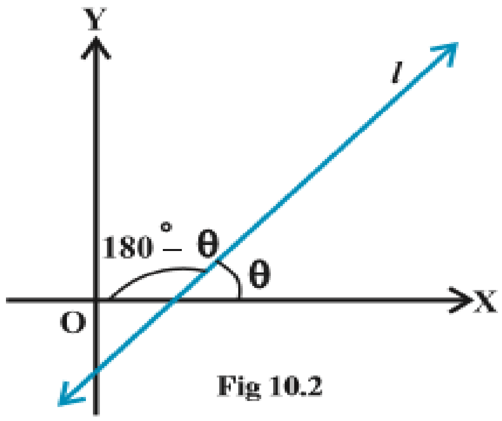

Fig 10.2

If **θ** is the inclination of the line with x-axis then '**tan θ**' is called the **slope** (or) gradient of the line and it is denoted as **'m'**

Thus , slope of a line **m = tan*θ***

Note: If **θ** = 0°, then the line is parallel to x-axis and if **θ** = 90°,then the line is perpendicular to x-axis (or) Parallel to y-axis.

Slope of a line when the any two points of the line is given

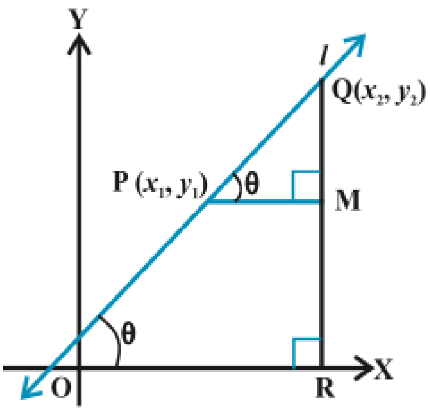

Fig 10. 3 (i)

Consider the above diagram,

Let $P(x_1,y_1)$ and $Q(x_2,y_2)$ on the line 'l' and let θ be the inclination of the line with x-axis. Draw the perpendiculars QR to x-axis and PM to QR. From the diagram, $\lfloor MBQ = \theta$.

Hence , Slope of the line l = m = tan $\theta = \frac{MQ}{MP} = \frac{y_2 - y_1}{x_2 - x_1}$

Simillarly, when θ is an obtuse angle we can find the slope as the same way.

Angle between two straight lines (in terms of slopes)

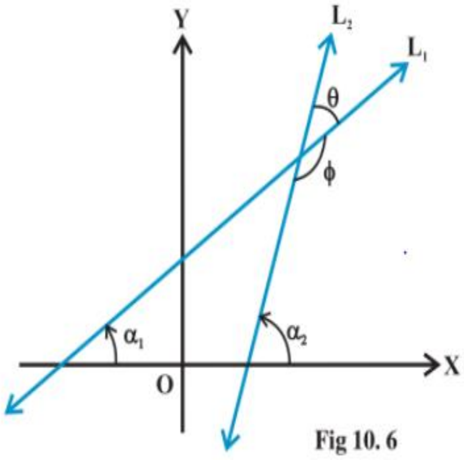

Fig 10.6

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2. If $\alpha 1$ *and* $\alpha 2$ are the inclinations of the lines then $m_1 = \tan \alpha 1$ and $m_2 = \tan \alpha 2$

We know that, when two lines intersect each other, they make a pair of vertically opposite angles such that their sum is 180^o . If θ *and* φ be the adjacent angles then

θ = $\alpha 2 - \alpha 1$ and $\varphi = 180 - \theta$

Now, tan**θ** = tan ($\alpha 2 - \alpha 1$)

$$= \frac{tan\alpha_2 - tan\alpha_1}{1+tan\alpha_1.tan\alpha_2} = \frac{m_2-m_1}{1+m_1.m_2} \text{ as } 1+m_1m_2 \neq 0 \text{---------(i)}$$

Also, $\tan \varphi = \tan(180 - \theta)$

$$= -\tan\theta = -\frac{m_2-m_1}{1+m_1.m_2}$$

i,e, $\tan\varphi = -\frac{m_2-m_1}{1+m_1.m_2}$ --------------------(ii)

Thus from (i) and (ii) we conclude that If tan**θ** is positive then $\tan\varphi$ is negative which means when θ *is acute* then φ will be obtuse and vice-versa.

Thus we get the formula for angle between two lines as

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1.m_2}\right| \text{ as } 1+m_1m_2 \neq 0.$$

This will always give the acute angle between two lines. We can find the other angle (obtuse) by using $180 - \theta$

Conditions for parallel and perpendicular lines

We have, $\tan\theta = \left|\frac{m_1-m_2}{1+m_1.m_2}\right|$

Case(i): If the two lines are parallel, then the angle between the lines $\theta = 0$

So, $\frac{m_1-m_2}{1+m_1.m_2} = \tan 0 = 0$

$\Rightarrow m_1 - m_2 = 0$

$\Rightarrow \boldsymbol{m_1 = m_2}$

This is the condition for two lines to be parallel.

Case(ii): If the two lines are perpendicular, then the angle between the lines $\theta = 90^o$

So, $\frac{m_1 - m_2}{1 + m_1.m_2} = \tan 90 = \infty$

$\Rightarrow 1 + m_1.m_2 = 0$

$\Rightarrow \boldsymbol{m_1.m_2 = -1}$

This is the condition for two lines to be perpendicular to each other

Collinearity of three points

We know that slope of two parallel lines are always equal. If two lines have the same slope and passes through a common point,then the two lines will coincide.

Moreover, if three points are collinear then the area of the triangle will become zero.

i.e, $\frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| = 0$

$\Rightarrow x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

Example Problems

Example-1. If the distance between the two points (a,-2) and (5,1) is 5 units, find the value(s) of a.

Solution: The distance between (a,-2) and (5,1)

$= \sqrt{(5-a)^2 + (1-(-2)^2}$

According to the given problem,

$\sqrt{(5-a)^2 + (1-(-2)^2} = 5$

$\Rightarrow \sqrt{(5-a)^2 + (3)^2} = 5$

$\Rightarrow (5-a)^2 + (3)^2 = 25$

$\Rightarrow (5-a)^2 = 25 - 9 = 16$

$\Rightarrow 5 - a = \pm 4$

$\Rightarrow a = 1, 9$

Hence, the required values of a are 1,9

Example:2. Find the ratio in which the point P whose abscissa is 3 divides the join of A(6,5) and B(-1,4). Hence, find the coordinates of P.

Solution: Let P divides the segment AB in the ratio k : 1
Hence by section formula,
the coordinates of P are $\left(\frac{-k+6}{k+1}, \ \frac{4k+5}{k+1}\right)$
But abscissa of point P is 3 (given)
So, $\frac{-k+6}{k+1} = 3 \ \Rightarrow 3k + 3 = -k + 6$
$\Rightarrow 4k = 3 \Rightarrow k = \frac{3}{4}$
Hence the required ratio is $\frac{3}{4}$:1, i.e, 3:4 internally
$\therefore$ Coordinate of P are $\left(\frac{\frac{-3}{4}+6}{\frac{3}{4}+1}, \frac{4\frac{3}{4}+5}{\frac{3}{4}+1}\right) = \left(\frac{21}{7}, \frac{32}{7}\right) = \left(3, \frac{32}{7}\right)$.

Example :3- If the points A(0,4),B(1,2) and C(3,3) are three corners of a square, find
(i)The coordinate of the point at which, the diagonals intersect.
(ii) The coordinate of D, the fourth corner of the square.

Solution: Let the fourth corner D be (a , b)

Now, AB= $\sqrt{(1-0)^2 + (2-4)^2} = \sqrt{1+4} = \sqrt{5}$
BC= $\sqrt{(3-1)^2 + (3-2)^2} = \sqrt{4+1} = \sqrt{5}$
AC= $\sqrt{(3-0)^2 + (3-4)^2} = \sqrt{9+1} = \sqrt{10}$
We note that, AB = BC and $AC^2 = AB^2 + BC^2$
i.e, 10 = 5 + 5
So ABCD is a square and BD is another diagonal

(i)Since diagonals of square bisect at the mid-point

Point of intersection = midpoint of AC = $\left(\frac{0+3}{2}, \frac{4+3}{2}\right) = \left(\frac{3}{2}, \frac{7}{2}\right)$

(ii) mid point of AC = mid-point of BD

$\left(\frac{0+3}{2}, \frac{4+3}{2}\right) = \left(\frac{a+1}{2}, \frac{b+2}{2}\right)$
$\Rightarrow \frac{a+1}{2} = \frac{3}{2}$ and $\frac{b+2}{2} = \frac{7}{2}$
$\Rightarrow a = 2$ and $b = 5$

So the fourth corner is (2,5)

Example-4: Find the area of the triangle whose vertices are (10,-6),(2,5) and (-1,3)

Solution: The area of the triangle whose vertices are (10,-6),(2,5) and (-1,3)

$= \frac{1}{2}|10(5-3)+2(3+6)+(-1)(-6-5)|$

$= \frac{1}{2}|20+18+11| = \frac{49}{2}$ sq.units.

Example 5: If the vertices of the triangle are (1,k),(4,-3) and (-9,7) and its area is 15 sq. units , find the value(s) of k.

Solution: Area of the triangle formed by the given points

$= \frac{1}{2}|1(-3-7)+4(7-k)+(-9)(k+3)|$

$= \frac{1}{2}|-10+28-4k-9k-27|$

$= \frac{1}{2}|-9-13k|$

As per the problem, $\frac{1}{2}|-9-13k| = \pm 15$

$\Rightarrow 9 + 13k = \pm 30 \Rightarrow 13k = -39 , 21$

$\Rightarrow k = -3, 21/13.$

Example 6: For what values of x are the points (1,5),(x,1) and (4,11) are collinear?

Solution: Since the points are collinear,

Area of the triangle formed by the points = 0

i.e, $\frac{1}{2}|1(1-11)+x(11-5)+(4)(5-1)| = 0$

$\Rightarrow |-10+6x+16| = 0$

$\Rightarrow 6x + 6 = 0$

$\Rightarrow x = -1$

Hence, the required value of x is -1

Example 7: Find the angle between the lines joining the points (-1,2) ,(3,-5) and (-2,3),(5,0).

Solution: Slope of the line joining (-1,2) ,(3,-5) is

$m_1 = \frac{-5-2}{3+1} = \frac{-7}{4}$

Slope of the line joining (-2,3),(5,0)is

$m_2 = \frac{0-3}{5+2} = \frac{-3}{7}$

Let $'\theta'$ be the angle between the given lines, then

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 . m_2}\right|$

$= \left|\frac{-\frac{7}{4} - (\frac{-3}{7})}{1 + (-\frac{7}{4})(\frac{-3}{7})}\right| = \left|\frac{-\frac{37}{28}}{\frac{7}{4}}\right| = \frac{37}{49}$

Hence the acute angle between the lines is given by $\tan\theta = \frac{37}{49}$

Problems for Practice

Exercise 10.1 complete from NCERT text book for class XI Mathematics