From this follow the results

(i) $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0}, \ \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \mathbf{0}, \ \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$

(ii) $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$

Note that the magnitude of $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is $\sin 90^0$ or 1, since $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ both have unit magnitude and the angle between them is 90^0. Thus, $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is a unit vector. A unit vector perpendicular to the plane of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ and related to them by the right hand screw rule is $\hat{\mathbf{k}}$. Hence, the above result. You may verify similarly,

$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$

From the rule for commutation of the cross product, it follows:

$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \ \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \ \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$

Note if $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ occur cyclically in the above vector product relation, the vector product is positive. If $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ do not occur in cyclic order, the vector product is negative.

Now,

$$\mathbf{a} \times \mathbf{b} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}) \times (b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}})$$

$$= a_x b_y \hat{\mathbf{k}} - a_x b_z \hat{\mathbf{j}} - a_y b_x \hat{\mathbf{k}} + a_y b_z \hat{\mathbf{i}} + a_z b_x \hat{\mathbf{j}} - a_z b_y \hat{\mathbf{i}}$$

$$= (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

We have used the elementary cross products in obtaining the above relation. The expression for $\mathbf{a} \times \mathbf{b}$ can be put in a determinant form which is easy to remember.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Example 7.4 *Find the scalar and vector products of two vectors.* $\mathbf{a} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ *and* $\mathbf{b} = (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}})$

Answer

$$\mathbf{a} \cdot \mathbf{b} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \cdot (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}})$$
$$= -6 - 4 - 15$$
$$= -25$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -4 & 5 \\ -2 & 1 & -3 \end{vmatrix} = 7\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

Note $\mathbf{b} \times \mathbf{a} = -7\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ ◀

7.6 ANGULAR VELOCITY AND ITS RELATION WITH LINEAR VELOCITY

In this section we shall study what is angular velocity and its role in rotational motion. We have seen that every particle of a rotating body moves in a circle. The linear velocity of the particle is related to the angular velocity. The relation between these two quantities involves a vector product which we learnt about in the last section.

Let us go back to Fig. 7.4. As said above, in rotational motion of a rigid body about a fixed axis, every particle of the body moves in a circle,

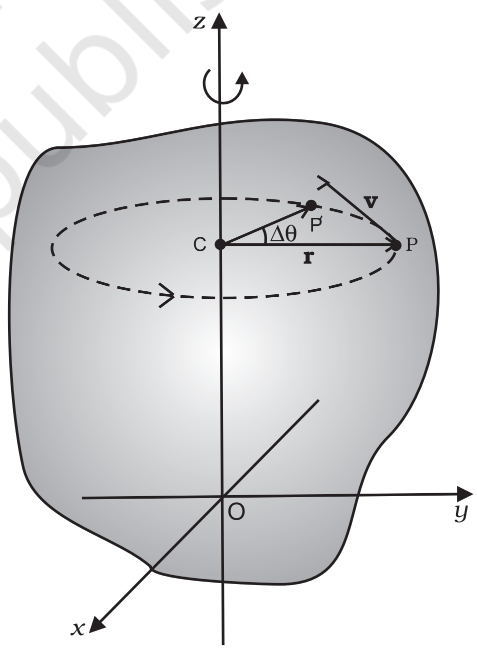

Fig. 7.16 *Rotation about a fixed axis. (A particle (P) of the rigid body rotating about the fixed (z-) axis moves in a circle with centre (C) on the axis.)*

which lies in a plane perpendicular to the axis and has its centre on the axis. In Fig. 7.16 we redraw Fig. 7.4, showing a typical particle (at a point P) of the rigid body rotating about a fixed axis (taken as the *z*-axis). The particle describes a circle with a centre C on the axis. The radius of the circle is r, the perpendicular distance of the point P from the axis. We also show the linear velocity vector $\mathbf{v}$ of the particle at P. It is along the tangent at P to the circle.

Let P′ be the position of the particle after an interval of time Δt (Fig. 7.16). The angle PCP′ describes the angular displacement $\Delta\theta$ of the particle in time Δt. The average angular velocity of the particle over the interval Δt is $\Delta\theta/\Delta t$. As Δt tends to zero (i.e. takes smaller and smaller values), the ratio $\Delta\theta/\Delta t$ approaches a limit which is the instantaneous angular velocity $d\theta/dt$ of the particle at the position P. We denote the **instantaneous angular velocity** by ω (the Greek letter omega). We know from our study of circular motion that the magnitude of linear velocity υ of a particle moving in a circle is related to the angular velocity of the particle ω by the simple relation $\upsilon = \omega r$, where r is the radius of the circle.

We observe that at any given instant the relation $\upsilon = \omega r$ applies to all particles of the rigid body. Thus for a particle at a perpendicular distance r_i from the fixed axis, the linear velocity at a given instant υ_i is given by

$$\upsilon_i = \omega r_i \tag{7.19}$$

The index i runs from 1 to n, where n is the total number of particles of the body.

For particles on the axis, $r = 0$, and hence $\upsilon = \omega r = 0$. Thus, particles on the axis are stationary. This verifies that the axis is *fixed*.

Note that we use the same angular velocity ω for all the particles. **We therefore, refer to $\boldsymbol{\omega}$ as the angular velocity of the whole body**.

We have characterised pure translation of a body by all parts of the body having the same velocity at any instant of time. Similarly, we may characterise pure rotation by all parts of the body having the same angular velocity at any instant of time. Note that this characterisation of the rotation of a rigid body about a fixed axis is **just another way** of saying as in Sec. 7.1 that each particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has the centre on the axis.

In our discussion so far the angular velocity appears to be a scalar. In fact, it is a vector. We shall not justify this fact, but we shall accept it. For rotation about a fixed axis, the angular velocity vector lies along the axis of rotation, and points out in the direction in which a right handed screw would advance, if the head of the screw is rotated with the body. (See Fig. 7.17a).

The magnitude of this vector is $\omega = d\theta/dt$ referred as above.

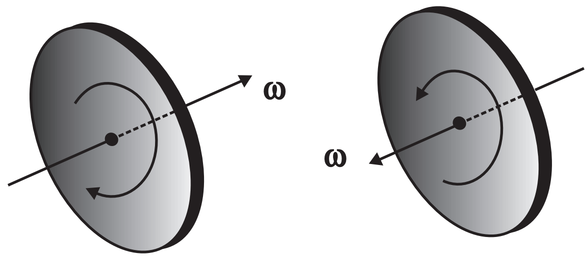

Fig. 7.17 (a) *If the head of a right handed screw rotates with the body, the screw advances in the direction of the angular velocity* $\boldsymbol{\omega}$. *If the sense (clockwise or anticlockwise) of rotation of the body changes, so does the direction of* $\boldsymbol{\omega}$.

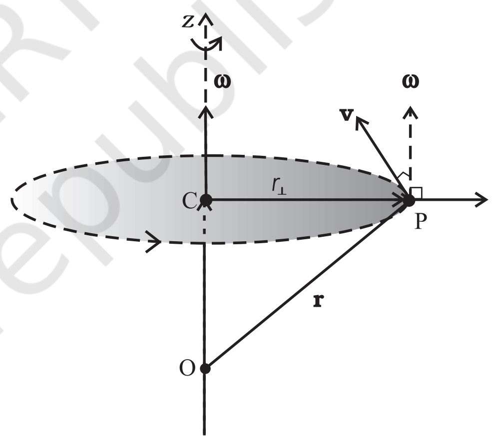

Fig. 7.17 (b) *The angular velocity vector* $\boldsymbol{\omega}$ *is directed along the fixed axis as shown. The linear velocity of the particle at P is* $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. *It is perpendicular to both* $\boldsymbol{\omega}$ *and* $\mathbf{r}$ *and is directed along the tangent to the circle described by the particle.*

We shall now look at what the vector product $\boldsymbol{\omega} \times \mathbf{r}$ corresponds to. Refer to Fig. 7.17(b) which is a part of Fig. 7.16 reproduced to show the path of the particle P. The figure shows the vector $\boldsymbol{\omega}$ directed along the fixed (z–) axis and also the position vector $\mathbf{r} = \mathbf{OP}$ of the particle at P of the rigid body with respect to the origin O. Note that the origin is chosen to be on the axis of rotation.

Now $\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{OP} = \boldsymbol{\omega} \times (\mathbf{OC} + \mathbf{CP})$

But $\boldsymbol{\omega} \times \mathbf{OC} = 0$ as $\boldsymbol{\omega}$ is along $\mathbf{OC}$

Hence $\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{CP}$

The vector $\boldsymbol{\omega} \times \mathbf{CP}$ is perpendicular to $\boldsymbol{\omega}$, i.e. to the z-axis and also to **CP**, the radius of the circle described by the particle at P. It is therefore, along the tangent to the circle at P. Also, the magnitude of $\boldsymbol{\omega} \times \mathbf{CP}$ is ω (CP) since $\boldsymbol{\omega}$ and **CP** are perpendicular to each other. We shall denote **CP** by $\mathbf{r}_\perp$ and not by **r**, as we did earlier.

Thus, $\boldsymbol{\omega} \times \mathbf{r}$ is a vector of magnitude $\omega r_\perp$ and is along the tangent to the circle described by the particle at P. The linear velocity vector **v** at P has the same magnitude and direction. Thus,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{7.20}$$

In fact, the relation, Eq. (7.20), holds good even for rotation of a rigid body with one point fixed, such as the rotation of the top [Fig. 7.6(a)]. In this case **r** represents the position vector of the particle with respect to the fixed point taken as the origin.

We note that **for rotation about a fixed axis, the direction of the vector ω does not change with time. Its magnitude may, however, change from instant to instant. For the more general rotation, both the magnitude and the direction of ω may change from instant to instant.**

7.6.1 Angular acceleration

You may have noticed that we are developing the study of rotational motion along the lines of the study of translational motion with which we are already familiar. Analogous to the kinetic variables of linear displacement (**s**) and velocity (**v**) in translational motion, we have angular displacement (**θ**) and angular velocity (**ω**) in rotational motion. It is then natural to define in rotational motion the concept of angular acceleration in analogy with linear acceleration defined as the time rate of change of velocity in translational motion. We define angular acceleration **α** as the time rate of change of angular velocity; Thus,

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} \tag{7.21}$$

If the axis of rotation is fixed, the direction of **ω** and hence, that of **α** is fixed. In this case the vector equation reduces to a scalar equation

$$\alpha = \frac{d\omega}{dt} \tag{7.22}$$

7.7 TORQUE AND ANGULAR MOMENTUM

In this section, we shall acquaint ourselves with two physical quantities (torque and angular momentum) which are defined as vector products of two vectors. These as we shall see, are especially important in the discussion of motion of systems of particles, particularly rigid bodies.

7.7.1 Moment of force (Torque)

We have learnt that the motion of a rigid body, in general, is a combination of rotation and translation. If the body is fixed at a point or along a line, it has only rotational motion. We know that force is needed to change the translational state of a body, i.e. to produce linear acceleration. We may then ask, what is the analogue of force in the case of rotational motion? To look into the question in a concrete situation let us take the example of opening or closing of a door. A door is a rigid body which can rotate about a fixed vertical axis passing through the hinges. What makes the door rotate? It is clear that unless a force is applied the door does not rotate. But any force does not do the job. A force applied to the hinge line cannot produce any rotation at all, whereas a force of given magnitude applied at right angles to the door at its outer edge is most effective in producing rotation. It is not the force alone, but how and where the force is applied is important in rotational motion.

The rotational analogue of force in linear motion is **moment of force**. It is also referred to as **torque** or **couple**. (We shall use the words moment of force and torque interchangeably.) We shall first define the moment of force for the special case of a single particle. Later on we shall extend the concept to systems of particles including rigid bodies. We shall also relate it to a change in the state of rotational motion, i.e. is angular acceleration of a rigid body.

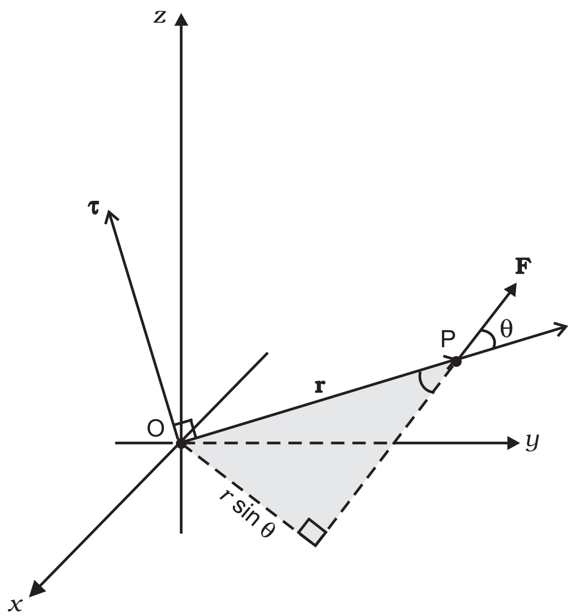

Fig. 7.18 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, *τ is perpendicular to the plane containing* **r** *and* **F**, *and its direction is given by the right handed screw rule.*

If a force acts on a single particle at a point P whose position with respect to the origin O is given by the position vector **r** (Fig. 7.18), the moment of the force acting on the particle with respect to the origin O is defined as the vector product

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \quad (7.23)$$

The moment of force (or torque) is a vector quantity. The symbol **τ** stands for the Greek letter *tau*. The magnitude of **τ** is

$$\tau = r F \sin\theta \quad (7.24a)$$

where r is the magnitude of the position vector **r**, i.e. the length OP, F is the magnitude of force **F** and θ is the angle between **r** and **F** as shown.

Moment of force has dimensions M $L^2 T^{-2}$. Its dimensions are the same as those of work or energy. It is, however, a very different physical quantity than work. Moment of a force is a vector, while work is a scalar. The SI unit of moment of force is newton metre (N m). The magnitude of the moment of force may be written

$$\tau = (r \sin\theta) F = r_\perp F \quad (7.24b)$$

$$\text{or } \tau = r F \sin\theta = r F_\perp \quad (7.24c)$$

where $r_\perp = r \sin\theta$ is the perpendicular distance of the line of action of **F** from the origin and $F_\perp (= F \sin\theta)$ is the component of **F** in the direction perpendicular to **r**. Note that $\tau = 0$ if $r = 0$, $F = 0$ or $\theta = 0^0$ or 180^0. Thus, the moment of a force vanishes if either the magnitude of the force is zero, or if the line of action of the force passes through the origin.

One may note that since $\mathbf{r} \times \mathbf{F}$ is a vector product, properties of a vector product of two vectors apply to it. If the direction of **F** is reversed, the direction of the moment of force is reversed. If directions of both **r** and **F** are reversed, the direction of the moment of force remains the same.

7.7.2 Angular momentum of a particle

Just as the moment of a force is the rotational analogue of force in linear motion, the quantity angular momentum is the rotational analogue of linear momentum. We shall first define angular momentum for the special case of a single particle and look at its usefulness in the context of single particle motion. We shall then extend the definition of angular momentum to systems of particles including rigid bodies.

Like moment of a force, angular momentum is also a vector product. It could also be referred to as moment of (linear) momentum. From this term one could guess how angular momentum is defined.

Consider a particle of mass m and linear momentum **p** at a position **r** relative to the origin O. The angular momentum ***l*** of the particle with respect to the origin O is defined to be

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} \quad (7.25a)$$

The magnitude of the angular momentum vector is

$$l = r p \sin\theta \quad (7.26a)$$

where p is the magnitude of **p** and θ is the angle between **r** and **p**. We may write

$$l = r p_\perp \text{ or } r_\perp p \quad (7.26b)$$

where $r_\perp$ (= $r \sin\theta$) is the perpendicular distance of the directional line of **p** from the origin and $p_\perp (= p \sin\theta)$ is the component of **p** in a direction perpendicular to **r**. We expect the angular momentum to be zero ($l = 0$), if the linear momentum vanishes ($p = 0$), if the particle is at the origin ($r = 0$), or if the directional line of **p** passes through the origin $\theta = 0^0$ or 180^0.

7.10.1 Theorem of parallel axes

This theorem is applicable to a body of any shape. It allows to find the moment of inertia of a body about any axis, given the moment of inertia of the body about a parallel axis through the centre of mass of the body. We shall only state this theorem and not give its proof. We shall, however, apply it to a few simple situations which will be enough to convince us about the usefulness of the theorem. The theorem may be stated as follows:

The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes. As shown in the Fig. 7.31, z and z' are two parallel axes, separated by a distance a. The z-axis passes through the centre of mass O of the rigid body. Then according to the theorem of parallel axes

$$I_{z'} = I_z + Ma^2 \qquad (7.37)$$

where I_z and $I_{z'}$ are the moments of inertia of the body about the z and z' axes respectively, M is the total mass of the body and a is the perpendicular distance between the two parallel axes.

Example 7.11 What is the moment of inertia of a rod of mass M, length l about an axis perpendicular to it through one end?

Answer For the rod of mass M and length l, $I = Ml^2/12$. Using the parallel axes theorem, $I' = I + Ma^2$ with $a = l/2$ we get,

$$I' = M\frac{l^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$$

We can check this independently since I is half the moment of inertia of a rod of mass $2M$ and length $2l$ about its midpoint,

$$I' = 2M.\frac{4l^2}{12} \times \frac{1}{2} = \frac{Ml^2}{3}$$

Example 7.12 What is the moment of inertia of a ring about a tangent to the circle of the ring?

Answer

The tangent to the ring in the plane of the ring is parallel to one of the diameters of the ring. The distance between these two parallel axes is R, the radius of the ring. Using the parallel axes theorem,

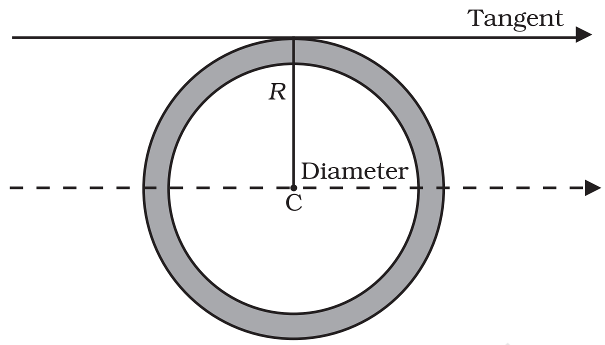

Fig. 7.32

$$I_{tangent} = I_{dia} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2.$$

7.11 KINEMATICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

We have already indicated the analogy between rotational motion and translational motion. For example, the angular velocity $\boldsymbol{\omega}$ plays the same role in rotation as the linear velocity $\mathbf{v}$ in translation. We wish to take this analogy further. In doing so we shall restrict the discussion only to rotation about fixed axis. This case of motion involves only one degree of freedom, i.e., needs only one independent variable to describe the motion. This in translation corresponds to linear motion. This section is limited only to kinematics. We shall turn to dynamics in later sections.

We recall that for specifying the angular displacement of the rotating body we take any particle like P (Fig.7.33) of the body. Its angular displacement θ in the plane it moves is the angular displacement of the whole body; θ is measured from a fixed direction in the plane of motion of P, which we take to be the x'-axis, chosen parallel to the x-axis. Note, as shown, the axis of rotation is the z – axis and the plane of the motion of the particle is the x - y plane. Fig. 7.33 also shows θ_0, the angular displacement at $t = 0$.

We also recall that the angular velocity is the time rate of change of angular displacement, $\omega = d\theta/dt$. Note since the axis of rotation is fixed, there is no need to treat angular velocity as a vector. Further, the angular acceleration, α = dω/dt.

The kinematical quantities in rotational motion, angular displacement (θ), angular velocity (ω) and angular acceleration (α) respectively are analogous to kinematic quantities in linear motion, displacement (x), velocity (v) and acceleration (a). We know the kinematical equations of linear motion with uniform (i.e. constant) acceleration:

$$v = v_0 + at \quad \text{(a)}$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \quad \text{(b)}$$

$$v^2 = v_0^2 + 2ax \quad \text{(c)}$$

where x_0 = initial displacement and v_0= initial velocity. The word 'initial' refers to values of the quantities at t = 0

The corresponding kinematic equations for rotational motion with uniform angular acceleration are:

$$\omega = \omega_0 + \alpha t \quad (7.38)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \quad (7.39)$$

$$\text{and } \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \quad (7.40)$$

where θ_0= initial angular displacement of the rotating body, and ω_0 = initial angular velocity of the body.

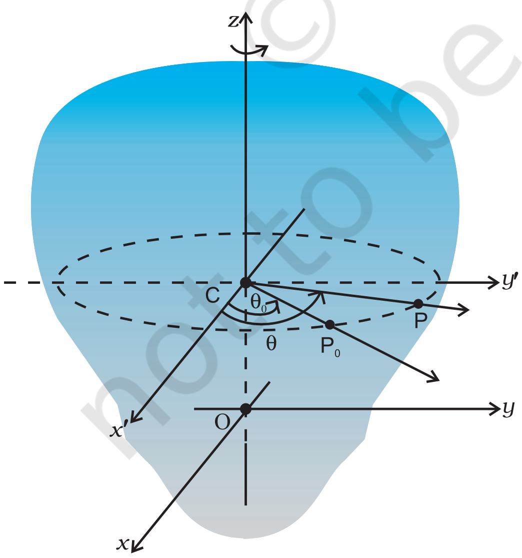

Fig.7.33 *Specifying the angular position of a rigid body.*

Example 7.13 Obtain Eq. (7.38) from first principles.

Answer The angular acceleration is uniform, hence

$$\frac{d\omega}{dt} = \alpha = constant \quad \text{(i)}$$

Integrating this equation,

$$\omega = \int \alpha \, dt + c$$

$$= \alpha t + c \quad \text{(as } \alpha \text{ is constant)}$$

At $t = 0$, $\omega = \omega_0$ (given)

From (i) we get at $t = 0$, $\omega = c = \omega_0$

Thus, $\omega = \alpha t + \omega_0$ as required.

With the definition of ω = dθ/dt we may integrate Eq. (7.38) to get Eq. (7.39). This derivation and the derivation of Eq. (7.40) is left as an exercise.

Example 7.14 The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds. (i) What is its angular acceleration, assuming the acceleration to be uniform? (ii) How many revolutions does the engine make during this time?

Answer

(i) We shall use $\omega = \omega_0 + \alpha t$

ω_0 = initial angular speed in rad/s

= 2π × angular speed in rev/s

$$= \frac{2\pi \times \text{angular speed in rev/min}}{60 \text{ s/min}}$$

$$= \frac{2\pi \times 1200}{60} \text{ rad/s}$$

$= 40\pi$ rad/s

Similarly ω = final angular speed in rad/s

$$= \frac{2\pi \times 3120}{60} \text{ rad/s}$$

= 2π × 52 rad/s

= 104 π rad/s

∴ Angular acceleration

$$\alpha = \frac{\omega - \omega_0}{t} = 4\pi \text{ rad/s}^2$$

The angular acceleration of the engine $= 4\pi$ rad/s^2

(ii) The angular displacement in time t is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= (40\pi \times 16 + \frac{1}{2} \times 4\pi \times 16^2) \text{ rad}$$

$$= (640\pi + 512\pi) \text{ rad}$$

$$= 1152\pi \text{ rad}$$

Number of revolutions $= \dfrac{1152\pi}{2\pi} = 576$ ◀

7.12 DYNAMICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

Table 7.2 lists quantities associated with linear motion and their analogues in rotational motion. We have already compared kinematics of the two motions. Also, we know that in rotational motion moment of inertia and torque play the same role as mass and force respectively in linear motion. Given this we should be able to guess what the other analogues indicated in the table are. For example, we know that in linear motion, work done is given by $F\,dx$, in rotational motion about a fixed axis it should be $\tau d\theta$, since we already know the correspondence $dx \rightarrow d\theta$ and $F \rightarrow \tau$. It is, however, necessary that these correspondences are established on sound dynamical considerations. This is what we now turn to.

Before we begin, we note *a* **simplification that arises in the case of rotational motion about a fixed axis**. Since the axis is fixed, only those components of torques, which are along the direction of the fixed axis need to be considered in our discussion. Only these components can cause the body to rotate about the axis. A component of the torque perpendicular to the axis of rotation will tend to turn the axis from its position. We specifically assume that there will arise necessary forces of constraint to cancel the effect of the perpendicular components of the (external) torques, so that the fixed position of the axis will be maintained. The perpendicular components of the torques, therefore need not be taken into account. This means that for our calculation of torques on a rigid body:

(1) We need to consider only those forces that lie in planes perpendicular to the axis. Forces which are parallel to the axis will give torques perpendicular to the axis and need not be taken into account.

(2) We need to consider only those components of the position vectors which are perpendicular to the axis. Components of position vectors along the axis will result in torques perpendicular to the axis and need not be taken into account.

Work done by a torque

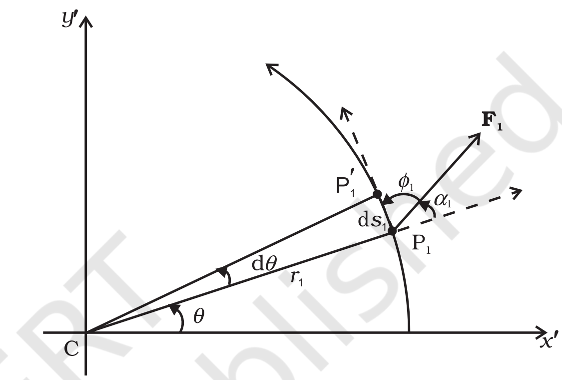

Fig. 7.34 *Work done by a force* $\mathbf{F}_1$ *acting on a particle of a body rotating about a fixed axis; the particle describes a circular path with centre C on the axis; arc* $P_1P'_1(ds_1)$ *gives the displacement of the particle.*

Figure 7.34 shows a cross-section of a rigid body rotating about a fixed axis, which is taken as the *z*-axis (perpendicular to the plane of the page; see Fig. 7.33). As said above we need to consider only those forces which lie in planes perpendicular to the axis. Let $\mathbf{F}_1$ be one such typical force acting as shown on a particle of the body at point P_1 with its line of action in a plane perpendicular to the axis. For convenience we call this to be the x'–y' plane (coincident with the plane of the page). The particle at P_1 describes a circular path of radius r_1 with centre C on the axis; $CP_1 = r_1$.

In time Δt, the point moves to the position P_1'. The displacement of the particle $d\mathbf{s}_1$, therefore, has magnitude $ds_1 = r_1 d\theta$ and direction tangential at P_1 to the circular path as shown. Here $d\theta$ is the angular displacement of the particle, $d\theta = \angle P_1CP_1'$. The work done by the force on the particle is

$$dW_1 = \mathbf{F}_1 . d\mathbf{s}_1 = F_1 ds_1 \cos\phi_1 = F_1(r_1\, d\theta)\sin\alpha_1$$

where ϕ_1 is the angle between $\mathbf{F}_1$ and the tangent