CLASS XII – MATHEMATICS
INTEGRATION

MODULE – 6/6

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DISTANCE LEARNING PROGRAMME : AN INITIATIVE BY AEES, MUMBAI

PREVIOUS KNOWLEDGE

- **TRIGONOMETRIC IDENTITIES**
- **DIFFERENTIATION**
- **STANDARD INTEGRATION FORMULAS**

PROPERTIES OF DEFINITE INTEGRALS – PROOFS

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$

Proof: substituting x= t, dx=dt we

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ **in particular** $\int_a^a f(x)dx = 0$

Proof: Let F(x) be anti-derivative of f(x)

$$\int_a^b f(x)dx = F(b) - F(a) = -(F(a) - F(b)) = -\int_b^a f(x)dx$$

If a= b we have $\int_a^b f(x)dx = F(a) - F(a) = 0$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Proof: Let F(x) be anti-derivative of f(x)

$$\int_a^b f(x)dx = F(b) - F(a) \text{.................. } (1)$$

$$\int_a^c f(x)dx = F(c) - F(a) \text{.................. } (2)$$

$$\int_c^b f(x)dx = F(b) - F(c) \text{.................. } (3)$$

Adding (2) and (3)

$$\int_a^c f(x)dx + \int_c^b f(x)dx = F(c) - F(a) + F(b) - F(c) = F(b) - F(a) = \int_a^b f(x)dx$$

4. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Proof: let x= a + b - t⇒ dx= - dt, when x=a t=b and when x=b t=a

Substituting in $\int_a^b f(x)dx$

$$\int_a^b f(x)dx = \int_b^a f(a+b-t)(-dt) = \int_a^b f(a+b-t)(dt) \text{ (by property 2)}$$

$$= \int_a^b f(a+b-x)(dx) \text{ (by property 1)}$$

5. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof: let a-x =t ⇒ dx= - dt, when x=0 t=a and when x=a t=0

Substituting in $\int_0^a f(x)dx = -\int_a^0 f(a-t)dt = \int_0^a f(a-t)dt$ (by property 2)

$= \int_0^a f(a-x)dx$ (by property 1)

6. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Proof: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$ .... (1)

In $\int_a^{2a} f(x)dx$, put 2a-x =t ⇒ dx= - dt, when x=a t=a and when x=2a t=0

Substituting in $\int_a^{2a} f(x)dx = -\int_a^0 f(2a-t)dt = \int_0^a f(2a-x)dx$ .... (2)

From (1) and (2)

$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

7. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x) \\ 0 \text{ if } f(2a-x) = -f(x) \end{cases}$

Proof: from property 6

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx \text{ .... } (1)$$

(i) if $f(2a-x) = f(x)$ .... (1) changes to

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$$

(ii) if $f(2a-x) = -f(x)$ .... (1) changes to

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$$

8. $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx \text{ if f is even function} \\ 0 \text{ if f is odd function} \end{cases}$

Proof: from property 3 $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$ .... (1)

Put $x = -t \Rightarrow dx = -dt$, when $x=-a$ $t=a$ and when $x=0$ $t=0$

Substituting in $\int_{-a}^{0} f(x)dx = -\int_a^0 f(-t)dt$

$$= \int_0^a f(-t)dt = \int_0^a f(-x)dx \text{ .... } (2)$$

From (1) and (2) $\int_{-a}^{a} f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx$ .... (2)

(i) if f is even function $f(-x) = f(x)$ $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ from equation 2

(ii) if f is odd function $f(-x) = -f(x)$ $\int_{-a}^{a} f(x)dx = 0$ from equation 2

EXAMPLES

By using the properties of definite integrals, evaluate the integrals

1. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}$ **2.** $\int_0^1 x(1-x)^n\,dx$ **3.** $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)dx$ **4.** $\int_2^8 |x-5|dx$

5. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\,dx$ **6.** $\int_0^{\pi} \log(1+\cos x)\,dx$

1. Find $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}$

Solution: $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}$ .... (1)

By using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad \text{.......................... (2)}$$

Adding (1) and (2)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = \int_0^{\frac{\pi}{2}} 1.\,dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0$$

$$2I = \frac{\pi}{2} \Longrightarrow I = \frac{\pi}{4}$$

Therefore, $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} = \frac{\pi}{4}$

2. **Find $\int_0^1 \mathbf{x(1-x)^n\,dx}$**

Solution: $\int_0^1 x(1-x)^n\,dx$

let $I = \int_0^1 x(1-x)^n\,dx$

I = $\int_0^1 (1-x)(1-(1-x))^n\,dx$,since $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

I = $\int_0^1 (1-x)(1-1+x)^n\,dx$ = $\int_0^1 (1-x)(x)^n\,dx$

= $\int_0^1 (x^n - x^{n+1})\,dx = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 = \frac{1}{n+1}$ - -$\frac{1}{n+2} = \frac{1}{(n+1)(n+2)}$

$\int_0^1 x(1-x)^n\,dx = \frac{1}{(n+1)(n+2)}$

Therefore, $\int_0^1 \mathbf{x(1-x)^n\,dx} = \frac{1}{(n+1)(n+2)}$

3. Find $\int_0^{\frac{\pi}{2}}(2\log\sin x - \log\sin 2x)dx$

Solution: Let I = $\int_0^{\frac{\pi}{2}}(2\log\sin x - \log\sin 2x)dx$

I = $\int_0^{\frac{\pi}{2}}(2\log\sin x - \log 2\sin x\cos x)dx$ since sin2x = 2sinxcosx

I = $\int_0^{\frac{\pi}{2}}(2\log\sin x - \log 2 - \log\sin x - \log\cos x)dx$

I = $\int_0^{\frac{\pi}{2}}(\log\sin x - \log 2 - \log\cos x)dx$ ....(1)

I = $\int_0^{\frac{\pi}{2}}\left(\log\sin(\frac{\pi}{2} - x) - \log 2 - \log\cos(\frac{\pi}{2} - x)\right)dx$ since $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

= $\int_0^{\frac{\pi}{2}}(\log\cos x - \log 2 - \log\sin x)dx$ .... (2)

Adding (1) and (2)

2I = $\int_0^{\frac{\pi}{2}}(\log\sin x - \log 2 - \log\cos x)dx + \int_0^{\frac{\pi}{2}}(\log\cos x - \log 2 - \log\sin x)dx$

= $\int_0^{\frac{\pi}{2}} -2\log 2\ dx$ = -2log2 $\int_0^{\frac{\pi}{2}} 1.dx$ = -2log2 $[x]_0^{\frac{\pi}{2}}$ = -2log2 ($\frac{\pi}{2}$ -0)

2I = -2log2 $\frac{\pi}{2} \Longrightarrow$ I = $-\frac{\pi}{2}$log2 = $\frac{\pi}{2}$ log $\frac{1}{2}$

Therefore, $\int_0^{\frac{\pi}{2}}(2\log\sin x - \log\sin 2x)dx = \frac{\pi}{2}\log\frac{1}{2}$

4. Find $\int_2^8 |x-5|dx$

Solution: Let I = $\int_2^8 |x-5|dx$

since (x-5) ≤ 0 for x ∈ [2,5] and x−5) ≥ 0 for x∈ [5,8]

we can write I = $-\int_2^5 (x-5)dx + \int_5^8 (x-5)dx$ (since $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$)

I = $\int_2^5 (-x+5)dx + \int_5^8 (x-5)dx = \left[\frac{-x^2}{2} + 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$

I = $\frac{-25}{2}$ +25 +2 -10 +32 -40 - $\frac{25}{2}$ +25 = 9

Therefore $\int_2^8 |x-5|dx = 9$

5. Find $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$

Solution: I = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$

$\sin^7 x = \sin^7(-x) = -\sin^7 x$ therefore $\sin^7 x$ is odd function, hence

I = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx = 0$ ($\int_{-a}^{a} f(x)dx = 0$ *if f is odd function*)

Therefore **$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx = 0$**

6. Find $\int_0^{\pi} \log(1+\cos x)\, dx$

Solution: Let I = $\int_0^{\pi} \log(1+cosx)\, dx$ .... (1)

I = $\int_0^{\pi} \log(1+\cos(\pi - x))\, dx$, since $\int_0^{a} f(x)dx = \int_0^{a} f(a-x)dx$

I = $\int_0^{\pi} \log(1-cosx)\, dx$ .... (2)

Adding (1) and (2)

2I = $\int_0^{\pi} \log(1+cosx)\, dx + \int_0^{\pi} \log(1-cosx)\, dx$

2I = $\int_0^{\pi} \log(1+cosx)(1-cosx)\, dx = \int_0^{\pi} \log(1-cos^2x)\, dx$

2I = $\int_0^{\pi} \log sin^2 x dx = \int_0^{\pi} 2logsinxdx$

I = $\int_0^{\pi} logsinxdx$ .... (A)

Since sin(π-x) = sinx

I = $\int_0^{\pi} logsinxdx = 2\int_0^{\frac{\pi}{2}} \log sinx\ dx$ ($\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ if $f(2a-x) = f(x)$)

I = $2\int_0^{\frac{\pi}{2}} \log sinx\ dx$ .... (3)

I = $2\int_0^{\frac{\pi}{2}} \log\sin(\frac{\pi}{2} - x)\ dx = 2\int_0^{\frac{\pi}{2}} \log\cos x\ dx$ .... (4)

Adding (3) and (4)

$2I = 2\int_0^{\frac{\pi}{2}} \log sinx\ dx + 2\int_0^{\frac{\pi}{2}} \log \cos x\ dx = 2\int_0^{\frac{\pi}{2}} \log \sin x \cos x\ dx$

$2I = = 2\int_0^{\frac{\pi}{2}} (\log 2\sin x \cos x - \log 2) dx = = 2\int_0^{\frac{\pi}{2}} (\log \sin 2x - \log 2)\ dx$

$I = \int_0^{\frac{\pi}{2}} (\log \sin 2x)\ dx + \int_0^{\frac{\pi}{2}} (-\log 2)\ dx$

Let $2x = t \Rightarrow 2dx = dt \Rightarrow dx = \frac{dt}{2}$ when x= 0, t=0 and when $x=\pi/2$, $t = 2\pi$

$I = \int_0^{\pi} logsint \frac{dt}{2} - (\log 2) \int_0^{\frac{\pi}{2}} dx$

$I = \frac{1}{2} \int_0^{\pi} logsint\ dt - \log 2\ [x]_0^{\frac{\pi}{2}}$

$I = \frac{1}{2} \int_0^{\pi} logsinx\ dx - log2\ (\frac{\pi}{2} - 0)$ (since $\int_a^b f(x)dx = \int_a^b f(t)dt$)

$I = \frac{1}{2} I - \frac{\pi}{2} \log 2$ from (A)

$I - \frac{1}{2} I = -\frac{\pi}{2} \log 2 \Rightarrow \frac{1}{2} I = -\frac{\pi}{2} \log 2$

Therefore, $\boldsymbol{\int_0^{\pi} \log(1 + \cos x)\ dx = -\pi \log 2}$

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