परमाणु ऊर्जा शिक्षण संस्था

(परमाणु ऊर्जा विभाग का स्वायत्त निकाय, भारत सरकार)

ATOMIC ENERGY EDUCATION SOCIETY

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Chapter 6.

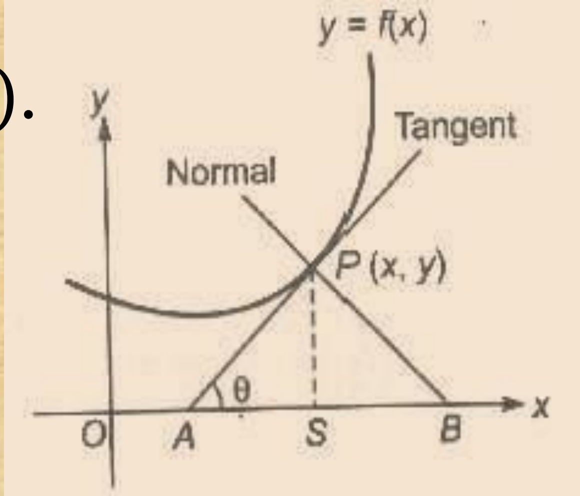
Application Of Derivatives

Module-2

e-content

Tangents And Normals

Recall in class XI we learned that $\frac{dy}{dx}$ (if it exists) geometrically it represents the slope of the tangent to the curve $y = f(x)$ at any point $P(x, y)$.



Thus, if $\theta(\neq \frac{\pi}{2})$ is the angle which the tangent to the curve at P makes with the positive direction of x-axis, then the slope of the tangent to the curve $y = f(x)$ at the point P=$\tan\theta = \frac{dy}{dx}$ $at\ P$.

If the tangent to the curve $y = f(x)$ at the point $P(x, y)$ is parallel to the x-axis, then $\theta = 0$.

$$\Rightarrow \tan\theta = 0 \Rightarrow \frac{dy}{dx} \ at\ P = 0.$$

If the tangent to the curve $y = f(x)$ at the point $P(x, y)$ is parallel to the y-axis, then $\theta = \frac{\pi}{2}$.

$$\Rightarrow \cot\ \theta = 0 \Rightarrow \frac{dx}{dy} \ at\ P = 0$$

Further, if the tangent at P is not parallel to X-axis i.e $\theta \neq 0\ i.e \tan\theta \neq \frac{dy}{dx}\ at\ P \neq 0,$ then the

Slope of normal to the curve at $P = -\frac{1}{\left(\frac{dy}{dx}\right)_P}$

(The normal is perpendicular to the tangent)

The gradient of a curve at a point is defined as the slope of the tangent to the curve at that point.

To find the equation of the tangent to the curve $\boldsymbol{y = f(x)}$ at a given point

- The given curve is $y = f(x)$
- Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then the slope of the tangent to the curve $y = f(x)$ at the point $P(x_1, y_1)$ is the value of $\frac{dy}{dx}$ $at\ P$.
- So, the slope of the tangent to the curve $y = f(x)$ at P is $\frac{dy}{dx}$ $at\ (x_1, y_1)$
- Therefore, by coordinate geometry, the equation of the tangent to the given curve $y = f(x)$ at the point $P(x_1, y_1)$ is
- $$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1).$$
- Remark:- If $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ does not exist, then the tangent at P is parallel to y-axis and its equation is $x = x_1$

Procedure to find the equation of the tangent to the curve $y = f(x)$ at the given point $P(x_1, y_1)$:-

- Find $\frac{dy}{dx}$ from the given equation $y = f(x)$.

- Find the value of $\frac{dy}{dx}$ at the given point $P(x_1, y_1)$, let $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$.

- The equation of the required tangent is $y - y_1 = m(x - x_1)$.

To find the equation of the normal to the curve $y = f(x)$ at a given point:-

- The given curve is $y = f(x)$
- Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then the slope of the tangent to the curve $y = f(x)$ at the point $P(x_1, y_1)$ is the value of $\frac{dy}{dx}\ at\ P$.
- So, the slope of the normal to the curve $y = f(x)$ at P is $P = -\frac{1}{\left(\frac{dy}{dx}\right)_P}$
- Therefore, by coordinate geometry, the equation of the tangent to the given curve $y = f(x)$ at the point $P(x_1, y_1)$ is
- $$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x - x_1).$$
- Remark:- If $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$, then the equation of the normal at P is $x = x_1$ and if $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ does not exist, then the equation of the normal at P is $y = y_{1.}$

Procedure to find the equation of the normal to the curve $y = f(x)$ at the given point $P(x_1, y_1)$:-

- Find $\frac{dy}{dx}$ from the given equation $y = f(x)$.
- Find the value of $\frac{dy}{dx}$ at the given point $P(x_1, y_1)$.
- If m is the slope of the normal to the given curve at P, then
- $m = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$
- The equation of the required normal is $y - y_1 = m(x - x_1)$.

Angle of intersection of two curves:-

- The angle of intersection of two curves is the angle between the tangents to the curves at their point of intersection.
- Let $y = f(x)\ and\ y = g(x)$ be the two given curves and P be a point of intersection of these two curves. Let m_1, m_2 be the slopes of the tangents at P to the two curves. If θ be the (acute) angle of intersection of the two curves at P, then
- $$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$
- In particular:-
- If $m_1 m_2 = -1$, the curves are said to cut orthogonally.
- If $m_1 = m_2$, the curves touch each other.

Example:-1. Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.

- **Solution:-** Let $P(x_1, y_1)$ be the required point.
- Given curve is $y = \sqrt{4x-3} - 1$ ………….(i)
- Differentiating (i) w.r.t.x, we get $\frac{dy}{dx} = (4x-3)^{\frac{-1}{2}}.4 = \frac{2}{\sqrt{4x-3}}$.
- $\therefore$ The slope of the tangent at $P = \frac{2}{\sqrt{4x_1-3}}$.
- According to given, $\frac{2}{\sqrt{4x_1-3}} = \frac{2}{3} \Rightarrow \sqrt{4x_1-3} = 3$
- $\Rightarrow 4x_1 - 3 = 9, \Rightarrow x_1 = 3$
- As $P(x_1, y_1)$ lies on (i), $y_1 = \sqrt{4x_1-3} - 1$
- $\Rightarrow \sqrt{4.3-3} - 1 = \sqrt{9} - 1 = 2$
- Hence, the required point is $(3, 2)$.

Example.2 Find the equations of the tangent and normal to the curve $16x^2 + 9y^2 = 145\ at\ (x_1, y_1)$, where $x_1 = 2\ and\ y_1 > 0$.

- **Solution:-** The given curve is $16x^2 + 9y^2 = 145$ ……………(i)
- Since (x_1, y_1) where $x_1 = 2, y_1 > 0$ lies on the curve (i), we get
- $16.2^2 + 9{y_1}^2 = 145 \Rightarrow 9{y_1}^2 = 145 - 64 \Rightarrow 9{y_1}^2 = 81$
- $\Rightarrow {y_1}^2 = 9 \Rightarrow y_1 = 3, -3\ but\ y_1 > 0 \ \Rightarrow y_1 = 3.$
- Therefore, the point $(x_1, y_1)\ is\ (2, 3)$.
- Differentiating (i) w.r.t.x, we get $16.2x + 9.2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{16x}{9y}$,
- $\therefore$ The slope of the tangent to the curve (i) at the point $(2, 3) = -\frac{16.2}{9.3} = -\frac{32}{27}$.

$\therefore$ The equation of the tangent to the curve at $(2, 3)$ *is*

$$y - 3 = -\frac{32}{27}(x - 2) \; or \; 32x + 27y = 145.$$

The slope of the normal to the given curve at the point (2, 3) is $\frac{27}{32}$ $\left(m_2 = -\frac{1}{m_1}\right)$

$\therefore$ The equation of the normal to the curve at $(2, 3)$ *is*

$$y - 3 = \frac{27}{32}(x - 2) \; or \; 27x - 32y + 42 = 0.$$

Example 3. Find all points on the curve $y = 4x^3 - 2x^5$ at which the tangents pass through origin.

- **Solution:-** The given curve is $y = 4x^3 - 2x^5$ …………(i)
- Let $P(x_1, y_1)$ be a point on the curve (i) at which the tangent to it pass through origin.
- Differentiating (i) w.r.t.x, we get,
- $\frac{dy}{dx} = 4.3x^2 - 2.5x^4 = 12x^2 - 10x^4$
- The slope of tangent to (i) at $P(x_1, y_1) = 12{x_1}^2 - 10{x_1}^4$
- The equation of tangent to the curve at $(x_1, y_1)\ is$
- $y - y_1 = (12{x_1}^2 - 10{x_1}^4)\ (x - x_1)$

- As it passes through origin (0, 0)
- $0 - y_1 = (12{x_1}^2 - 10{x_1}^4)\,(0 - x_1)$
- $\Rightarrow y_1 = x_1(12{x_1}^2 - 10{x_1}^4)$……………….(ii)
- As the given curve passes through
- $P(x_1, y_1),\ y_1 = 4{x_1}^3 - 2{x_1}^5$……………….(iii)
- From (ii) and (iii), we get
- $4{x_1}^3 - 2{x_1}^5 = 12{x_1}^3 - 10{x_1}^5 \Rightarrow 8\,{x_1}^5 - 8{x_1}^3 = 0$
- $\Rightarrow {x_1}^3({x_1}^2 - 1) = 0\ \ \Rightarrow x_1 = 0, 1, -1$
- When $x_1 = 0, y_1 = 0;\quad when\ x_1 = 1, y_1 = 2;\ when$
- $x_1 = -1,\ y_1 = -2$
- Hence, the required points are $(0, 0), (1, 2), (-1, -2)$.
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THANK YOU

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